Quadratic duality and Koszul duality – Tim

<u>Setup:</u> $k = \bar{k}$, $A = \bigoplus_{i \in \mathbb{N}} A_i$ positively graded k-alg

<u>Assume:</u> A_0 semisimple, in fact $A_0 \cong \underbrace{k \times \dots \times k}_{n\text{-times}}$

A_i f.d. v.sp.

A-mod := cat of finite dim'l A-modules

A-gmod := graded A-modules, f.d. graded pieces, morphisms preserve degrees

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<u>Def'n:</u> Set $m: A_1 \otimes A_1 \longrightarrow A_2$ the multiplication of A

$$m^*: A_2^* \longrightarrow (A_1 \otimes A_1)^* \cong A_1^* \otimes A_1^*$$

Then $A^! := T(A_1^*)/_{\operatorname{im} m^*}$ <u>quadratic dual</u> of A

If A is quadratic, i.e. $A \cong T(A_1)/(R)$ for $R \subset A_1 \otimes_{A_0} A_1$

then $A^! \cong T(A_1^*)/(R^\perp)$ and $(A^!)^! \cong A$

<u>Example:</u> $A = \mathbb{C}[x] \Rightarrow A^! = \mathbb{C}[x]/(x^2)$

<u>Goal:</u> Relate the representation theory of A and $A^!$

<u>Recall:</u> A is called <u>Koszul</u> if A_0 has a <u>linear</u> projective resolution in A-gmod

$$\dots \longrightarrow P^{-1} \longrightarrow P^0 \twoheadrightarrow A_0$$

"<u>linear</u>": P^{-m} is generated in deg m

In this case: A is quadratic, $A^!$ is Koszul, $(A^!)^{op} \cong \operatorname{Ext}_{A\text{-mod}}(A^0, A^0)$

"Koszul dual"

<u>Source:</u> Mazorchuk – Ovsienko – Stroppel

Then, we will see, that

$A_Q^!$ is the path algebra of Q, but with the relation $ab=0$.

So, $A_Q \cong A_Q^!$. In fact, A_Q is Koszul algebra

- Indecomposable projective graded A-modules are given by

$$\underbrace{P(i)}\langle m\rangle = A e_i \langle m\rangle$$

Note: $\langle . \rangle$ means shift in grading

later: $[.]$ means shift in homol. degree

$$\left[P(i) = \mathcal{C}(i,-) = \{ \mathcal{C} \longrightarrow \underset{A\text{-gmod}}{g\text{-vect}_k} \} \right]$$

- simple graded A-modules

1-dim'l

$$L(i)\langle m\rangle = \left(P(i) / P(i)_{>0} \right) \langle m\rangle$$

$$\left[\begin{array}{rcl} L(i) : & \mathcal{C} \longrightarrow & \mathrm{Vect} \\ & i \longmapsto & k \\ & 0 \neq j \longmapsto & 0 \end{array} \right]$$

in example 1: $L(1) = \mathbb{C}$,

$L^!(1) = \mathbb{C}$

- indecomposable injective graded A-modules:

Nakayama functor

$N : A\text{-gmod} \longrightarrow A\text{-gmod}$

$M \longmapsto A^* \otimes_A M$

$$I(i) := N(P(i))\langle m\rangle = A^* e_i \langle m\rangle$$

in example 1: $I(1) \cong \mathbb{C}[x]^*$

$$I^!(1) \cong \left(\mathbb{C}[x]/(x^2) \right)^* \cong \mathbb{C}[x]/(x^2)\langle -1\rangle$$

Prop: $LCP(A)$ is an abelian category.

Pf: It suffices to check:

| if $X \xrightarrow{f} Y$ is a morphism in $LCP(A)$, then
| $\ker_{C(A)}(f)$ and $\operatorname{coker}_{C(A)}(f)$ are in $LCP(A)$.

- $X^m \xrightarrow{f^m} Y^m$ is a sum of morphisms $P(i)\langle m\rangle \longrightarrow P(j)\langle m\rangle$
- $$\operatorname{Hom}_{A\text{-gmod}}(P(i)\langle m\rangle, P(j)\langle m\rangle) = \begin{cases} 0 & \text{if } i \neq j \\ k & \text{if } i = j \end{cases}$$

b/c $Ae_i\langle m\rangle \xrightarrow{\Phi} Ae_j\langle m\rangle$ is determined by $\Phi(e_i) = \lambda e_j$. If $i \neq j$, $\underset{\Phi(e_i)}{\underset{\|}{\Phi(e_i^2)}} = e_i\lambda e_j = 0$

If $i = j$, $\Phi(\alpha) = \lambda\alpha$.

$\Rightarrow$ everything in here is 0 or an isomorphism. ∎

Note:

$$\begin{tikzcd}
C(A\text{-gmod}) \arrow[r] & K(A\text{-gmod}) \arrow[r] & D(A\text{-gmod}) \\
LCP(A) \arrow[u, phantom, "\cup"] \arrow[r, "\simeq"] & LCP(A) \arrow[u, phantom, "\cup"] \arrow[r, "\simeq"] & LCP(A) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

homotopy category

since we restricted to minimal complexes

This gives an answer to our main question:

(if A is Koszul)

$$\begin{tikzcd}
D^{\uparrow}(A\text{-gmod}) \arrow[r, phantom, "\simeq"] & D^{\downarrow}(A^!\text{-gmod}) \\
\boxed{LCP(A)} \arrow[u, phantom, "\cup"] & A^!\text{-gmod} \arrow[u, phantom, "\cup"] \arrow[l, "\simeq"']
\end{tikzcd}$$

(*)

Then $\mathcal{I}_i^\bullet := S Q_i^\bullet$ (proj resol of $L(i)$) respectively $\mathcal{P}_i^\bullet := S N^{-1} I_i^\bullet$

are an injective hull resp. a projective cover of $\mathcal{L}_i$ and these are all indecomposable inj / proj. up to a shift.

<u>In our examples:</u>

1) $A = \mathbb{C}[x]$ $\qquad A^! = \mathbb{C}[x]/(x^2)$

$A^!$-gmod

$L^!(1) = \mathbb{C}$

$I^!(1) \cong \mathbb{C}[x]/(x^2)\langle -1\rangle$

$P^!(1) = \mathbb{C}[x]/(x^2)$

LCP(A)

$$\begin{array}{llllll}
 & & -1 & 0 & 1 & 2 \\
\mathcal{L}_1^\bullet : & 0 \longrightarrow & 0 \longrightarrow & \mathbb{C}[x] \longrightarrow & 0 \longrightarrow & 0 \\
\mathcal{I}_1^\bullet & 0 \longrightarrow & \mathbb{C}[x]\langle 1\rangle \xrightarrow{\cdot x} & \mathbb{C}[x] \longrightarrow & 0 \longrightarrow & 0 \\
\mathcal{P}_1^\bullet & 0 \longrightarrow & 0 \longrightarrow & \mathbb{C}[x] \xrightarrow{\cdot x} & \mathbb{C}[x]\langle -1\rangle \longrightarrow &
\end{array}$$

uses proj resol of $\mathbb{C}$ as $\mathbb{C}[x]$-mod:

$$\mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x] \twoheadrightarrow \mathbb{C}$$

2) $A = A_Q$ $\quad (ba = 0)$ $\qquad A^! : ab = 0$

$A^!$-gmod

$L^!(1) = \langle e_1\rangle$

$L^!(2) = \langle e_2\rangle$

$I^!(2) = \langle a^*, e_2\rangle$

$I^!(1) \cong P^!(1)\langle -2\rangle$

$P^!(2) = \langle e_2, b\rangle$

$P^!(1) = \langle e_1, a, ba\rangle$

LCP(A)

$\mathcal{L}_1^\bullet : \; 0 \longrightarrow 0 \longrightarrow P(1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$

$\mathcal{L}_2^\bullet : \; 0 \longrightarrow 0 \longrightarrow P(2) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$

$\mathcal{I}_2^\bullet : \; 0 \longrightarrow P(1)\langle 1\rangle \xrightarrow{\cdot b} P(2) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$

$\mathcal{I}_1^\bullet \; P(1)\langle 2\rangle \xrightarrow{\cdot b} P(2)\langle 1\rangle \xrightarrow{\cdot a} P(1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$

$\mathcal{P}_2^\bullet \; 0 \longrightarrow 0 \longrightarrow P(2) \xrightarrow{\cdot a} P(1)\langle -1\rangle \longrightarrow 0 \longrightarrow$

$\mathcal{P}_1^\bullet \; 0 \longrightarrow 0 \longrightarrow P(1) \xrightarrow{\cdot b} P(2)\langle -1\rangle \xrightarrow{\cdot a} P(1)\langle -2\rangle$

Example:

$\mathfrak{g} = \mathfrak{sl}_2$. $\mathcal{O}_0$ = simple modules $L(0)$, $L(-2)$

finite dim'l ∞-dim'l

$$M(-2) \cong L(-2)$$

$$M(0)$$

$$M(-2) \cong L(-2) \hookrightarrow M(0) \twoheadrightarrow L(0)$$

linear rep

projectives: $P(0)$

$P(-2) \cong M(-1) \otimes L(1)$

$M(0) \hookrightarrow P(-2) \twoheadrightarrow M(-2)$

$P = P(0) \oplus P(-2)$ is a proj. generator

$\mathrm{End}_{\mathcal{O}}(P) \cong A_Q$ from before.